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An alternative approach for solving Maxwell equations

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Abstract

At present the use of hypercomplex methods is pursued by a growing number of mathematicians, physicists and engineers. Quaternionic and Clifford calculus will be applied on wide classes of problems in very different fields of science. We explain Maxwell equations within the geometric algebras of real and complex quaternions. The connection between Maxwell equations and the Dirac equation will be elaborated. Using the Teodorescu transform we will deduce an iteration procedure for solving weak time-dependent Maxwell equations in isotropic homogeneous media. Assuming the so-called Drude-Born-Feodorov constitutive laws Maxwell equations in chiral media were deduced. Full time-dependent problems will be reduced to the consideration of Weyl operators.

1 Historical oriented introduction

Classical Maxwell equations were discovered in the second half of the nineteenth century as result of the stormy development of electromagnetic research in that time. The study of these equations has attracted generations of physicists and mathematicians but some of their secrets are still hidden.

At about the same time, also new algebraic structures were invented. W.R. Hamilton discovered in 1843 the algebra of real quaternions as a generalization of the field of complex numbers. Under the influence of H. Grassman's extension theory and Hamilton's quaternions, W.K. Clifford created in 1878 a geometric algebra, which is nowadays called Clifford algebra. Its construction starts with a basis in the signed $R^n = R^{p,q}$ with units e_1, \dots, e_n . Assume that $e_i^2 = -1$, for $i = 1, \dots, q$, and $e_j^2 = 1$, for $j = 1, \dots, p$, as well as the anticommutator relation

$$e_i e_j + e_j e_i = 0$$

for $i \neq j$. Together with $e_0 = 1$ one can construct a basis in the 2^n -dimensional standard Clifford algebra $Cl_{p,q}$. Incidentally, in 1954 C. Chevalley [5] showed that each Clifford number, i.e. each element of $Cl_{p,q}$, can be identified with an antisymmetric tensor.

Let us go back to the electromagnetic field equations. Already J. C. Maxwell [15] himself and W. R. Hamilton [10] used these new algebraic techniques to try to simplify

Maxwell's equations. The aim was to obtain an equation of the type

$$Du + au = F$$

with suitable operators D and a . For this reason Hamilton introduced his "N'abla operator" as well as the notion "vector". The tendency of algebraisation of physics continued in the first half of the last century. A long list of important publications were devoted to this topic. We only stress here some of the milestones, beginning with the "Theory of Relativity" by L. Silberstein (1914)[18], and H. Weyl's book "Raum-Zeit-Materie" of 1921. Important results of Einstein/Mayer, Lanczos and Proca followed. In 1935 this development highlighted with the thesis of M. Mercier (Genève) [16]. After the reinvention of the concept of "spinors", firstly appeared in 1911 in a paper by E. Cartan, D. Hestenes [11, 12, 13], F. Bolinder [3] and M. Riesz [17] wrote fundamental algebra papers with applications in electromagnetic theory, using the framework of Clifford numbers and spinor spaces.

Meanwhile, in the late thirties the famous Swiss mathematician R. Fueter and his co-workers and followers used a function-theoretic approach for the same problems. These ideas were refreshed and fruitful extended by R. Delanghe and his group and A. Sudbury in the seventies and early eighties (cf. [4, 20]). Influenced by the success of complex analysis and Vekua theory a generalized operator theory with corresponding singular integral operators [19] and a corresponding hypercomplex theory for boundary value problems of elliptic partial differential equations were developed [8],[9].

Making use of a transformation of Maxwell's equations into a system of homogeneous coordinates we will propose an alternative solution method.

2 Maxwell equations

Let G be a bounded domain with sufficient smooth boundary Γ that is filled out with an isotropic homogeneous material.

Using Gauss units Maxwell equations read as follows:

$$\begin{aligned} c \operatorname{rot} H &= 4\pi J + \partial_t D && \text{(Biot-Savart-Ampere's law)} \\ c \operatorname{rot} E &= -\partial_t B && \text{(Faraday's law)} \\ \operatorname{div} D &= 4\pi\rho && \text{(Coulomb's law)} \\ \operatorname{div} B &= 0 && \text{(no free magnetic charge)} \end{aligned}$$

Furthermore, the continuity condition has to be fulfilled:

$$\operatorname{div} J = -\partial_t \rho,$$

where $E = E(t, x)$ is the electric field, $H = H(t, x)$ the magnetic field, $J = J(t, x)$ the electric current density, $D = D(t, x)$ the electric flux density, $B = B(t, x)$ the magnetic flux density, $\rho = \rho(t, x)$ the charge density, and c is the speed of light in a vacuum.

The relations between flux densities and the electric and magnetic fields depend on the material. It is well-known that for instance all organic materials contain carbon and

realize in this way some kind of optical activity. Therefore, Lord Kelvin introduced the notion of the chirality measure of a medium. This coefficient expresses the optical activity of the underlying material. The correspondent constitutive laws are the following:

$$\begin{aligned} D &= \epsilon E + \epsilon \beta \operatorname{rot} E && \text{(Drude-Born-Feodorov laws),} \\ B &= \mu H + \mu \beta \operatorname{rot} H, \end{aligned}$$

where $\epsilon = \epsilon(t, x)$ is the electric permittivity, $\mu = \mu(t, x)$ is the magnetic permeability and the coefficient β describes the chirality measure of the material. In isotropic cases one has the possibility to use the so-called Tellegen representation

$$\begin{aligned} D &= \epsilon E + \alpha H, \\ B &= \mu H + \alpha^* E. \end{aligned}$$

The connection between the electric field E and current density J is given by

$$J = \sigma E + \sigma g$$

where σ is the electric conductivity and g a given electric source.

Starting with $\beta = 0$ and replacing D and B by $D = \epsilon E$ and $B = \mu H$ we get in the case of

$$\epsilon = \epsilon(x), \mu = \mu(x)$$

$$-\epsilon \partial_t E + c \operatorname{rot} H = 4\pi J, \quad (2.1)$$

$$\mu \partial_t H + c \operatorname{rot} E = 0, \quad (2.2)$$

$$\epsilon \operatorname{div} E = 4\pi \rho - (\nabla \epsilon \cdot E), \quad (2.3)$$

$$\mu \operatorname{div} H = -(\nabla \mu \cdot H). \quad (2.4)$$

After summing (2.1) and (2.4) as well as (2.2) and (2.3) we obtain

$$-\epsilon \partial_t E + c \operatorname{rot} H + \mu \operatorname{div} H = -(\nabla \mu \cdot H) + 4\pi J, \quad (2.5)$$

$$\mu \partial_t H + c \operatorname{rot} E + \epsilon \operatorname{div} E = -(\nabla \epsilon \cdot H) + 4\pi \rho. \quad (2.6)$$

In the case of ϵ, μ being constants we can introduce the new functions \tilde{E}, \tilde{H} which are defined on a homogeneous space with a first coordinate x_0 and the other coordinates $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$. We obtain:

$$E(t, x) =: \tilde{E} \left(-\frac{1}{\epsilon} t, \frac{1}{c} \tilde{x} \right),$$

$$H(t, x) =: \tilde{H} \left(\frac{1}{\mu} t, \frac{1}{c} \tilde{x} \right).$$

The equations (2.5)–(2.6) transform into

$$\partial_1 \tilde{E} + \operatorname{rot} \tilde{H} + \mu c \operatorname{div} \tilde{H} = 4\pi J,$$

$$\partial_1 \tilde{H} + \operatorname{rot} \tilde{E} + \epsilon c \operatorname{div} \tilde{E} = 4\pi \rho.$$

3 Quaternionic representations

Let e_1, e_2, e_3 be the generating units of the algebra of real quaternions \mathbb{H} , which fulfil the conditions

$$e_i e_j + e_j e_i = -2\delta_{ij} \quad (i, j = 1, 2, 3).$$

This leads to the following multiplication rule for two quaternions $u = u_0 + \underline{u}$, $v = v_0 + \underline{v}$:

$$uv = u_0 v_0 - \underline{u} \cdot \underline{v} + u_0 \underline{v} + v_0 \underline{u} + \underline{u} \times \underline{v} \quad (v_i \in \mathbb{R}),$$

where $\underline{u} = u_1 e_1 + u_2 e_2 + u_3 e_3$, $\underline{v} = v_1 e_1 + v_2 e_2 + v_3 e_3$. Further, let $u = u_0 + \underline{u}$ be a quaternion. Then $\bar{u} = u_0 - \underline{u}$ is called to be its **conjugate quaternion**. The operator defined by

$$D = \partial_1 e_1 + \partial_2 e_2 + \partial_3 e_3$$

is called **Dirac operator**. It acts on a quaternionic valued function as follows:

$$Du = -\operatorname{div} \underline{u} + \operatorname{rot} \underline{u} + \operatorname{grad} u_0.$$

With the multiplication operator m_θ

$$m_\theta u = \theta u_0 + \underline{u} \quad (\theta \in \mathbb{R}^+),$$

with $u = u_0 + \underline{u}$, $\underline{u} = u_1 e_1 + u_2 e_2 + u_3 e_3$, we obtain

$$m_{\mu c}(\partial_1 \tilde{E} + D \tilde{H}) = 4\pi J,$$

$$m_{\epsilon c}(\partial_1 \tilde{H} + D \tilde{E}) = 4\pi \rho,$$

and so

$$\partial_1 \tilde{E} + D \tilde{H} = m_{\mu c}^{-1} 4\pi J,$$

$$\partial_1 \tilde{H} + D \tilde{E} = m_{\epsilon c}^{-1} 4\pi \rho.$$

Finally, we get

$$\partial(\tilde{E} + \tilde{H}) = \partial_1(\tilde{E} + \tilde{H}) + D(\tilde{E} + \tilde{H}) = 4\pi(m_{\mu c}^{-1} J + m_{\epsilon c}^{-1} \rho) =: F_1,$$

$$\overline{\partial}(\tilde{E} - \tilde{H}) = \partial_1(\tilde{E} - \tilde{H}) - D(\tilde{E} - \tilde{H}) = 4\pi(m_{\mu c}^{-1} J - m_{\epsilon c}^{-1} \rho) =: F_2,$$

where ∂ is also called **Weyl operator** and $\overline{\partial}$ is the conjugate to ∂ . By the way, a function u is called **quaternionic regular** if $\partial u = 0$ and **quaternionic anti-regular** if $\overline{\partial} u = 0$.

For simplifying we set: $\tilde{E} + \tilde{H} =: v$ and $\tilde{E} - \tilde{H} =: w$. Then it follows

$$\partial w = F_1(v, w), \tag{3.1}$$

$$\overline{\partial} w = F_2(v, w). \tag{3.2}$$

Let us have a closer look at the functions F_1, F_2 . The electric current density J is given by

$$J = \sigma E + \sigma g,$$

where E and g are vector functions. This leads to the following simplification

$$F_1 = 4\pi \left[\sigma(\tilde{E} + g) + \frac{\rho}{\varepsilon c} \right] = 2\pi \left[\sigma(v + w) + \sigma g + \frac{\rho}{\varepsilon c} \right],$$

$$F_2 = 4\pi \left[\sigma(\tilde{E} + \sigma g) - \frac{\rho}{\varepsilon c} \right] = 2\pi \left[\sigma(v + w) + g - \frac{\rho}{\varepsilon c} \right].$$

Hence

$$F_2 = -\bar{F}_1.$$

Thus

$$\frac{\partial w}{\partial} = F_1(v, w), \quad (3.3)$$

$$\bar{\partial} w = -\bar{F}_1(v, w). \quad (3.4)$$

4 Integral representation

Let G be a bounded domain in \mathbb{R}^3 and a a positive constant. We consider in \mathbb{R}^4 the cylinder $Z = G \times [-a, a]$. A right inverse to the Weyl operator is the following **Teodorescu transform**:

$$(T_Z u)(x) = \frac{-1}{\sigma_3} \int_Z e(x-y) u(y) dy, \quad Z = G \times [-a, a]$$

with $e(x) = \bar{x}/|x|^4$, $\sigma_3 = 2\pi^{3/2}/\Gamma(3/2)$. We obtain in a straightforward manner

$$\partial T_Z u = \begin{cases} u & \text{in } Z, \\ 0 & \text{in } \mathbb{R}^4 \setminus \bar{Z}, \end{cases}$$

and

$$T_Z \partial u + \phi_Z = \begin{cases} u & \text{in } Z, \\ 0 & \text{in } \mathbb{R}^4 \setminus \bar{Z}, \end{cases}$$

with $\phi_Z \in \ker \partial$. In complete analogy a conjugate Teodorescu transform T_Z^* is introduced. We just have to replace $e(x)$ by its conjugate. Now it follows from (7)–(8) that

$$\begin{aligned} v &= T_Z F_1(v, w) + \phi_Z \quad (\partial \phi_Z = 0), \\ w &= T_Z^* F_2(v, w) + \phi_Z^* \quad (\bar{\partial} \phi_Z^* = 0). \end{aligned}$$

Furthermore we have to introduce Cauchy-Bizadse-type operators, which are defined by the boundary data. These operators read as follows:

$$(F_{\partial Z} u)(x) = \frac{1}{\sigma_3} \int_{\partial Z} e(x-y) n(y) u(y) d(\partial Z)_y \quad (x \notin \partial Z)$$

and

$$(F_{\partial Z}^* u)(x) := \frac{1}{\sigma_3} \int_{\partial Z} \bar{e}(x-y) n(y) u(y) d(\partial Z)_y, \quad (x \notin \partial Z)$$

where $n(y) = (n_0 + \underline{n})(y)$ denotes the unit vector of the outer normal on ∂Z at the point y .

It can be proved that

$$\phi_Z^* = F_{\partial Z}^* w \quad \text{and} \quad \phi_Z = F_{\partial Z} v \quad \text{in } Z.$$

It should be noted that we do not need the whole trace of the functions w and v on the boundary. We just have to consider these parts of $\text{tr}_Z v$ ($\text{tr}_Z w$) which are lying in the corresponding Hardy space of functions, which permit a quaternionic regular (quaternionic anti-regular) extension into Z , accordingly. We get the integral equations

$$v = 4\pi\sigma T_Z(v + w) + 4\pi T_Z(\sigma g + \frac{\rho}{\epsilon c}) + h, \quad (4.1)$$

$$w = 4\pi\sigma T_Z^*(v + w) + 4\pi T_Z^*(\sigma g - \frac{\rho}{\epsilon c}) + h^*, \quad (4.2)$$

where

$$h = F_{\partial Z} \text{tr}_{\partial Z} v \quad \text{and} \quad h^* = F_{\partial Z}^* \text{tr}_{\partial Z} w.$$

If h, h^* are known then under smallness conditions the iteration procedure:

$$v_n = 4\pi\sigma T_Z(v_{n-1} + w_{n-1}) + 4\pi T_Z(\sigma g + \frac{\rho}{\epsilon c}) + h,$$

$$w_n = 4\pi\sigma T_Z^*(v_{n-1} + w_{n-1}) + 4\pi T_Z^*(\sigma g - \frac{\rho}{\epsilon c}) + h^*,$$

with ($v_0 = w_0 = 0$) will converge in suitable Banach spaces.

Remark 4.1 In [1] is proved the following estimation:

$$\|T_Z\|_{L(L_\infty, C)} \leq \frac{2\sigma_3^2}{3} a |G|.$$

5 Weak time dependent Maxwell-equations

Assume now $\epsilon = \epsilon(x)$, $\mu = \mu(x)$, $\kappa = \kappa(x)$ ($g = 0$) and

$$E(t, x) = E_0(t)E(x) \quad \text{and} \quad H(t, x) = H_0(t)H_1(x),$$

where the scalar functions E_0 and H_0 are known. Maxwell equations then transform to

$$c E_0 \cdot \text{rot} E_1 = -\partial_t(\mu H_0)H_1, \quad (5.1)$$

$$c H_0 \cdot \text{rot} H_1 = (\partial_t(\epsilon E_0) + 4\pi\kappa E_0)E_1, \quad (5.2)$$

$$E_0(\nabla \epsilon \cdot E_1) + \epsilon \text{div} E_1 = 4\pi\rho, \quad (5.3)$$

$$(\nabla \mu \cdot H_1) + \mu \text{div} H_1 = 0. \quad (5.4)$$

It follows

$$\text{rot} E_1 = -\frac{\mu}{c} \frac{\partial_t H_0}{E_0} H_1 =: \alpha_0 H_1,$$

$$\text{rot} H_1 = \left(\frac{\epsilon}{c} \frac{\partial_t E_0}{H_0} + \frac{4\pi\kappa}{c} \frac{E_0}{H_0} \right) E_1 =: \beta_0 E_1,$$

$$-\text{div} E_1 = -\frac{4\pi\rho}{\epsilon E_0} + \frac{\nabla \epsilon}{\epsilon} \cdot E_1 = \rho' - \underline{\alpha} \cdot E_1,$$

$$-\operatorname{div} H_1 = \frac{\nabla \mu}{\mu} \cdot H_1 = -\underline{\beta} \cdot H_1.$$

Here $\alpha = \alpha_0 + \underline{\alpha}$, $\beta = \beta_0 + \underline{\beta}$, $\alpha := -\frac{\nabla \varepsilon}{\varepsilon}$, $\beta := -\frac{\nabla \mu}{\mu}$. Using the fact that in \mathbb{H}

$$D\underline{u} = -\operatorname{div} \underline{u} + \operatorname{rot} \underline{u},$$

we get

$$\begin{aligned} D E_1 &= \alpha_0 H_1 + \rho' - \underline{\alpha} \cdot E_1, \\ D H_1 &= \beta_0 E_1 - \underline{\beta} \cdot H_1. \end{aligned}$$

The right inverse of D is the corresponding Teodorescu transform T_G over $G \subset \mathbb{R}^3$. A short calculation leads to

$$\begin{aligned} E_1 &= T_G \alpha_0 H_1 - T_G \underline{\alpha} \cdot E_1 + T_G \rho' + \phi_1, \\ H_1 &= T_G \beta_0 E_1 - T_G \underline{\beta} \cdot H_1 + \phi_2, \end{aligned}$$

where $\phi_i \in \ker D$ ($i = 1, 2$). The iteration method

$$\begin{aligned} E_1^{(n)} &= -T_G \underline{\alpha} \cdot E_1^{(n-1)} + T_G \alpha_0 H_1^{(n-1)} + T_G \rho' + \phi_1, \\ H_1^{(n)} &= T_G \underline{\beta} \cdot E_1^{(n)} - T_G \underline{\beta} \cdot H_1^{(n-1)} + \phi_2, \end{aligned}$$

with $H_1^{(0)} = E_1^{(0)} = 0$ converges in suitable Banach spaces (L_2, W_2^1, C) under smallness conditions.

In the time-harmonic case i.e. $H_0 = E_0 \equiv 1$ and ε, μ , are constants and $\kappa = \kappa(x)$ we have

$$DE_1 = \rho' \quad \text{and} \quad DH_1 = \beta_0 E_1.$$

Setting $\beta_0 = \delta^{-1}$ we obtain

$$D \delta D H_1 = \frac{4\pi\rho}{\varepsilon} = \rho',$$

i.e.

$$\Delta H_1 = -f.$$

If boundary values of H_1 ($\operatorname{tr}_\Gamma H_1$) are known i.e. $\operatorname{tr}_\Gamma H_1 = g$ the complete solution is given by

$$H_1 = F_\Gamma g + T_G \mathcal{P}_\delta D h + T_G \mathcal{Q}_\delta \delta T_G f. \quad (5.5)$$

Here \mathcal{P}_δ and \mathcal{Q}_δ are orthoprojections on subspaces in the quaternionic Hilbert space $L_2(G)$, namely

$$L_2(G) = \delta \ker D \cap L_2(G) \bigoplus_{\delta} {}^\circ D W_2^1(G).$$

The scalar product is defined by

$$(u, v)_\delta := \int_G \bar{u} \delta v dG \in \mathbb{H}.$$

The operator \mathcal{P}_δ can be seen as a generalized **Bergman projection**.

In the representation formula from above is F_Γ the Cauchy-Bizadse operator on Γ and h a smooth continuation of g into G . Note that \mathcal{P}_δ and \mathcal{Q}_δ can be explicitly defined (cf. [9])! Then

$$E_1 = \frac{c}{4\pi\kappa} \mathcal{P}_\delta Dh + \mathcal{Q}_\delta \delta T_G f.$$

Let us prove that the boundary condition is fulfilled! Indeed,

$$\mathcal{Q}_\delta T_G f = D\tilde{f} \quad \text{with } \tilde{f} \in \overset{\circ}{W}_2^1 \text{ i.e. } tr_\Gamma \tilde{f} = 0.$$

$$T_G D\tilde{f} = \tilde{f} - F_\Gamma \tilde{f} = 0 \quad (\text{Borel-Pompeiu's formula}).$$

On the other hand, Plemelj-Sokhotzkij's formulae yield:

$$\begin{aligned} tr_\Gamma H_1 &= P_\Gamma g + tr_\Gamma \mathcal{P}_\delta Dh = P_\Gamma g + tr_\Gamma TDh - tr_\Gamma T \mathcal{Q}_\delta Dh \\ &= P_\Gamma g + g - P_\Gamma g + 0 = g. \end{aligned}$$

P_Γ is the so-called Plemelj-projection onto that Hardy space of \mathbb{H} -regular extendible functions into G .

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